However, this particular sophisticated model utilized for the dynamic analysis of aerospace vehicles vessels with end caps, although having complicated numerical operations, offers accurate frequency results to be obtained with an high convergence rate.

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Classical Normal Modes in Nonviscously Damped Linear Systems

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Nomenclature

$\boldsymbol{C}$	=	viscous damping matrix
$\mathcal{G}(t)$	=	damping function in the time domain
$\boldsymbol{I}_N$	=	identity matrix of size N
$\boldsymbol{K}$	=	stiffness matrix
$\boldsymbol{M}$	=	mass matrix
N	=	degrees of freedom of the system
$\mathbb{R}$	=	space of real numbers
t	=	time
$\boldsymbol{U}, \boldsymbol{V}$	=	matrices of the right and left eigenvectors
$\mathbf{u}(t)$	=	generalized coordinates
α_i	=	set of real scalars
$\delta(t)$	=	Dirac-delta function
μ_1, μ_2	=	parameters of the GHM[5] damping model

Superscripts

$\bullet^T$	=	matrix transposition of $\bullet$
$\bullet^{-T}$	=	inversed transpose of $\bullet$
$\bullet^{-1}$	=	matrix inversion of $\bullet$
$\dot{\bullet}$	=	differentiation of $\bullet$ with respect to t

Introduction

IN general, dynamic systems are nonviscously damped. Possibly the most general way to model damping within the linear range is to consider nonviscous damping models that depend on the past history of motion via convolution integrals over some kernel functions. The equations of motion describing free vibration of an N-degree-of-freedom linear system with such damping can be expressed by

$$\boldsymbol{M}\ddot{\mathbf{u}}(t) + \int_{-\infty}^{t} \mathcal{G}(t-\tau)\dot{\mathbf{u}}(\tau)\,\mathrm{d}\tau + \boldsymbol{K}\mathbf{u}(t) = \mathbf{0} \tag{1}$$

where $\boldsymbol{M}, \boldsymbol{K}, \mathcal{G}(t) \in \mathbb{R}^{N\times N}$. In the special case when $\mathcal{G}(t-\tau) = \boldsymbol{C}\delta(t-\tau)$, Eq. (1) reduces to the case of viscously damped systems. A damping model of this kind is a further generalization of the familiar viscous damping. It is well known that under certain conditions viscously damped symmetric systems possess classical normal modes, that is, $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{C}$ can be diagonalized simultaneously by a real congruence transformation. The most common example in this regard is proportional damping, where the viscous damping matrix has the special form

$$\boldsymbol{C} = \alpha_1\boldsymbol{M} + \alpha_2\boldsymbol{K}, \qquad \alpha_1, \alpha_2 \in \mathbb{R} \tag{2}$$

This damping model is also known as Rayleigh damping or classical damping. Caughey and O'Kelly[1] have proved that viscously damped linear systems with symmetric coefficient matrices possess classical normal modes if, and only if, the relationship

$$\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{K} \tag{3}$$

is satisfied. Based on this result, they have shown that the series representation of damping

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$$C = M \sum_{j=0}^{N-1} \alpha_j [M^{-1}K]^j \tag{4}$$

is the necessary and sufficient condition for existence of classical normal modes. Later, Ma and Caughey[2] proved that, for the case when the system matrices are not symmetric, Eq. (3) still describes as the necessary and sufficient condition for simultaneous diagonalization of the system matrices by an equivalence transformation.

Because all of the preceding concepts are in the context of viscously damped systems, it is not clear whether such concepts exist for nonviscously damped systems of the form (1). The purpose of this Note is to address this issue. Specifically, we ask the following question: under what conditions can nonviscously damped systems be classically damped? That is, under what conditions can M, K and $\mathcal{G}(t)$ be simultaneously diagonalized? For the sake of generality, we consider that all of the system matrices are asymmetric.

Existence of Classical Normal Modes

It is required to find out the conditions when two nonzero matrices $U \in \mathbb{R}^{N\times N}$ and $V \in \mathbb{R}^{N\times N}$ exist such that they simultaneously diagonalize M, K, and $\mathcal{G}(t)$ under an equivalence transformation. Unlike the viscously damped case where all of the system matrices are constant, here the system dynamics is characterized by two constant matrices and one matrix containing real functions. The problem of simultaneous diagonalization of Hermitian matrices with functional entries through constant complex transformations has been discussed by Chakrabarti et al.[3] In view of their results and considering the system is positive definite, the conditions for simultaneous diagonalization of Eq. (1) can be described as follows.

Theorem 1: If M, K, and $\mathcal{G}(t)$, $\forall t$ are positive definite matrices and there exist two non-singular matrices $U \in \mathbb{R}^{N\times N}$ and $V \in \mathbb{R}^{N\times N}$ such that V^TMU, V^TKU, and $V^T\mathcal{G}(t)U$, $\forall t$ are all real diagonal matrices then the following are equivalent.

Condition 1:

$$KM^{-1}\mathcal{G}(t) = \mathcal{G}(t)M^{-1}K$$

Condition 2:

$$MK^{-1}\mathcal{G}(t) = \mathcal{G}(t)K^{-1}M$$

Condition 3:

$$M\mathcal{G}^{-1}(t)K = K\mathcal{G}^{-1}(t)M, \qquad \forall t$$

Proof: We have to prove that 1) from the given condition conditions 1–3 follow and 2) from conditions 1–3 the given condition follows, in total, all of the six statements proposed in the theorem. If $\mathcal{G}(t)$ is a sufficiently smooth matrix function, then one can obtain a sequence $\mathcal{G}_r = \mathcal{G}(t_r) \in \mathbb{R}^{N\times N}$, $\forall r = 1, 2, \ldots, \infty$, where all $\mathcal{G}_r$ are positive definite. For notational brevity, define $\mathfrak{Z} = \{M, \mathcal{G}_r, K\}$ as a ordered collection of the system property matrices. Instead of proving all of the six statements separately, first we will prove that there exist U and V such that $V^T\mathfrak{Z}_kU$ is a real diagonal if and only if there exist an i, $1 \le i \le 3$, such that $\mathfrak{Z}_j\mathfrak{Z}_i^{-1}\mathfrak{Z}_m = \mathfrak{Z}_m\mathfrak{Z}_i^{-1}\mathfrak{Z}_j$ for all $j, m = 1, 2, 3 \neq i$, and then will come back to our main result.

Consider the if part first: Let $\mathfrak{Z}_i$ be the positive definite matrix, then there exist two matrices L and R such that $L^T\mathfrak{Z}_iR = I$ or $\mathfrak{Z}_i^{-1} = RL^T$. From the given condition $\mathfrak{Z}_j\mathfrak{Z}_i^{-1}\mathfrak{Z}_m = \mathfrak{Z}_m\mathfrak{Z}_i^{-1}\mathfrak{Z}_j$, we have $\mathfrak{Z}_j(RV^T)\mathfrak{Z}_m = \mathfrak{Z}_m(RL^T)\mathfrak{Z}_j$ or $(L^T\mathfrak{Z}_jR)(L^T\mathfrak{Z}_mR) = (L^T\mathfrak{Z}_mR)(L^T\mathfrak{Z}_jR)$. This implies that $(L^T\mathfrak{Z}_jR)$, $\forall j$, are pairwise commutative matrices. Thus, from the classical theorem of simultaneous diagonalization (see Ref. 4, page 52), we say that there exist a nonzero S such that $S^{-1}(L^T\mathfrak{Z}_jR)S$ is diagonal $\forall j$. Thus, the if-part follows by selecting $U = RS$ and $V = LS^{-T}$.

To prove the only-if part, suppose $(V^T\mathfrak{Z}_iU) = \Lambda_i$ is a diagonal matrix with its elements $\lambda_{i_s} > 0$, $\forall s$. Therefore, $\Lambda_i^{-1/2}(V^T\mathfrak{Z}_iU)\Lambda_i^{-1/2} = I$, from which one has $\mathfrak{Z}_i = V^{-T}\Lambda_i^{1/2}\Lambda_i^{1/2}U^{-1}$ or

$$\mathfrak{Z}_i^{-1} = U\Lambda_i^{-\frac{1}{2}}\Lambda_i^{-\frac{1}{2}}V^T \tag{5}$$

Now from the given condition, $(V^T\mathfrak{Z}_jU) = \Lambda_j$ is a diagonal matrix $\forall j \neq i$, or $\Lambda_i^{-1/2}(V^T\mathfrak{Z}_jU)\Lambda_i^{-1/2} = \Lambda_{j/i}$ a diagonal matrix. A similar expression for $\Lambda_{m/i}$ can also be obtained considering that $(V^T\mathfrak{Z}_mU)$ is diagonal. Because two diagonal matrix always commute, we have $\Lambda_{j/i}\Lambda_{m/i} = \Lambda_{m/i}\Lambda_{j/i}$, $\forall j, m \neq i$, or

$$\Lambda_i^{-\frac{1}{2}}V^T\mathfrak{Z}_jU\Lambda_i^{-\frac{1}{2}}\Lambda_i^{-\frac{1}{2}}V^T\mathfrak{Z}_mU\Lambda_i^{-\frac{1}{2}}$$
$$= \Lambda_i^{-\frac{1}{2}}V^T\mathfrak{Z}_mU\Lambda_i^{-\frac{1}{2}}\Lambda_i^{-\frac{1}{2}}V^T\mathfrak{Z}_jU\Lambda_i^{-\frac{1}{2}}$$

Using the expression of $\mathfrak{Z}_i^{-1}$ obtained before in Eq. (5) results in $\mathfrak{Z}_j\mathfrak{Z}_i^{-1}\mathfrak{Z}_m = \mathfrak{Z}_m\mathfrak{Z}_i^{-1}\mathfrak{Z}_j$, $\forall j, m \neq i$.

Because in this Proof l, m, and i are all arbitrary and $\mathfrak{Z}_i^{-1}$ exist, $\forall i = 1, 2, 3$, all of the six statements proposed in the theorem have been proved by successive change of indices. □

Note that condition 1 of this theorem is analogous to the result obtained by Ma and Caughey[2] (Theorem 3, Ref. 2) on viscously damped systems. If the system is not positive definite, then the results of Theorem 1 still describe the necessary and sufficient conditions for simultaneous diagonalization; however, in such cases, U and V in general are complex matrices. Further, Theorem 1 can also be applied to systems with singular matrices. If a system is $\bullet$ singular, then the condition(s) involving $\bullet^{-1}$ have to be disregarded and the remaining condition(s) should be used.

Generalization of the Proportional Damping

In this section the concept of proportional damping is generalized to nonviscously damped systems. Consider the conditions 1 and 2 of Theorem 1; premultiplying condition 1 by M^{-1} and condition 2 by K^{-1}, one has

$$A\mathcal{B}(t) = \mathcal{B}(t)A, \qquad A^{-1}\mathcal{D}(t) = \mathcal{D}(t)A^{-1} \qquad \forall t \tag{6}$$

where $A = M^{-1}K$, $\mathcal{B}(t) = M^{-1}\mathcal{G}(t)$, and $\mathcal{D}(t) = K^{-1}\mathcal{G}(t)$. It is well known that for any two matrices A and B, if A commutes with B, $f(A)$ also commutes with B where $f(z)$ is any analytic function of the variable z. Thus, representations like $M^{-1}\mathcal{G}(t) = \mathcal{F}(M^{-1}K, t)$ and $K^{-1}\mathcal{G}(t) = \mathcal{F}(K^{-1}M, t)$ are valid for any $\mathcal{F}(z, t)$ analytic in z. Adding these two quantities and also taking A and A^{-1} in the argument of the function as (trivially) A and A^{-1} always commute, we can express the damping function matrix in the form of

$$\mathcal{G}(t) = M\mathcal{F}_1(M^{-1}K, K^{-1}M, t) + K\mathcal{F}_2(M^{-1}K, K^{-1}M, t) \tag{7}$$

such that the system possesses classical normal modes. Furthermore, postmultiplying condition 1 of Theorem 1 by M^{-1} and condition 2 by K^{-1}, one obtains $(KM^{-1})(\mathcal{G}(t)M^{-1}) = (\mathcal{G}(t)M^{-1})(KM^{-1})$ and $(MK^{-1})(\mathcal{G}(t)K^{-1}) = (\mathcal{G}(t)K^{-1})(MK^{-1})$, respectively. Following a similar procedure, from these relationships we can express the damping function matrix in the form

$$\mathcal{G}(t) = \mathcal{F}_3(KM^{-1}, MK^{-1}, t)M + \mathcal{F}_4(KM^{-1}, MK^{-1}, t)K \tag{8}$$

for which system (1) possesses classical normal modes. The functions $\mathcal{F}_i(z_1, z_2, t)$, $i = 1, 2, 3, 4$, can have any general forms as long as they are analytic in z_1 and z_2. Although the functions $\mathcal{F}_i$, $i = 1, 2, 3, 4$, are general, the expression of $\mathcal{G}(t)$ in Eqs. (7) or (8) gets restricted because of the special nature of the arguments in the functions.

The proportional damping [in Eq. (2)] can be obtained directly from Eq. (7) or (8) as a very special, one could almost say trivial, case by choosing each matrix function as $\mathcal{F}_i = \alpha_i\delta(t)I_N$. The damping functions expressed in Eq. (7) or (8) provides a new way of interpreting the proportional damping or classical damping where the identity matrices (always) associated in the right- or left-hand side of M and K are replaced by arbitrary matrix functions $\mathcal{F}_i$ with proper arguments. This kind of damping will be called generalized proportional damping. We call the representation in Eq. (7) right-functional form and that in Eq. (8) left-functional form. The Caughey series [in Eq. (4)] is an example of right functional form with the time functions as the delta function. From this discussion, we say that nonviscously damped positive definite systems possess classical normal modes if and only if $\mathcal{G}(t)$ can be represented by Eq. (7) or (8). If the system matrices are not positive definite, then

Eq. (7) or (8) provides a sufficient condition for the existence of classical normal modes. Furthermore, if $\boldsymbol{M}$ or $\boldsymbol{K}$ is singular, then the arguments involving its corresponding inverse have to be removed from the functions to use these expressions.

Example

Consider an asymmetric system of form (1) whose mass and stiffness matrices are given by[2]

$$\boldsymbol{M} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Suppose that the damping mechanism of the system is double exponential (GHM[5]) so that the matrix of damping functions has the form

$$\mathcal{G}(\hat{\tau}) = \begin{bmatrix} 0.6e^{-\mu_1\hat{\tau}} + 1.368e^{-\mu_2\hat{\tau}} & -0.1e^{-\mu_1\hat{\tau}} - 0.368e^{-\mu_2\hat{\tau}} \\ -0.5e^{-\mu_1\hat{\tau}} - e^{-\mu_2\hat{\tau}} & 0 \end{bmatrix} \tag{9}$$

where $\hat{\tau} = t - \tau$ and μ_1, μ_2 are real positive constants. Note that none of the system matrices are positive definite; moreover, $\boldsymbol{K}$ is singular.

The matrices of undamped right and and left eigenvectors are, respectively,[2]

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{V} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Because conditions 1 and 3 of Theorem 1 are satisfied, the system can be decoupled by an equivalence transformation involving $\boldsymbol{U}$ and $\boldsymbol{V}$. One easily verifies that $\boldsymbol{V}^T\boldsymbol{M}\boldsymbol{U} = \text{diag}[1, -1]$, $\boldsymbol{V}^T\boldsymbol{K}\boldsymbol{U} = \text{diag}[0, -1]$, and

$$\boldsymbol{V}^T\mathcal{G}(t)\boldsymbol{U} = \begin{bmatrix} 0.5e^{-\mu_1\hat{\tau}} + e^{-\mu_2\hat{\tau}} & 0.0 \\ 0.0 & -0.1e^{-\mu_1\hat{\tau}} - 0.638e^{-\mu_2\hat{\tau}} \end{bmatrix}$$

is a diagonal matrix $\forall\hat{\tau}$. Note that $\mathcal{G}(\hat{\tau})$ given in Eq. (9) can be represented in terms of $\boldsymbol{M}$ and $\boldsymbol{K}$ as $\mathcal{G}(\hat{\tau}) = \boldsymbol{M}e^{-(\boldsymbol{M}^{-1}\boldsymbol{K})^2/2}\cos(\boldsymbol{M}^{-1}\boldsymbol{K})e^{-\mu_1\hat{\tau}} + e^{-\sqrt{(\boldsymbol{K}\boldsymbol{M}^{-1})}}\boldsymbol{M}e^{-\mu_2\hat{\tau}}$. This also illustrates the applicability of the generalized proportional damping proposed here.

Conclusions

Conditions for the existence of classical normal modes in nonviscously damped asymmetric linear multiple-degree-of-freedom systems have been derived. The nonviscous damping mechanism is such that it depends on the past history of the velocities via convolution integrals over some kernel functions. By introducing the concept of generalized proportional damping, we have extended the applicability of classical damping. The generalized proportional damping expresses the damping in terms of any nonlinear function involving time and specially arranged mass and stiffness matrices so that the system still posses classical normal modes. This enables analysis of a more general class of nonviscously damped asymmetric discrete linear dynamic systems using classical modal analysis.

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